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TITLE- Optimization of a Very-Low Capacity  
Channel Using a Multi-Tone Frequency  
Shift Keyed Detector - Part II

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AUTHOR(S)- L. Schuchman

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ABSTRACT

This paper extends the investigation of lunar-Earth, planetary-Earth, and deep space low capacity communication channels initiated by the author in a previous memorandum titled "Optimization of a Very-Low Capacity Channel Using a Multi-Tone Frequency Shift Keyed Detector." (TM-69-2034-4, May 5, 1969)

Two models of phase noise (whose significance is that it is the defining characteristic of low capacity channels) and their corresponding derived maximum likelihood detectors are discussed. The two detectors are defined as the random step detector and the random linear drift detector. It is argued that the "optimum" detector will depend upon the degree of knowledge one has about the statistics and character of the phase noise and the ability to take advantage of this information to produce a realizable and practical detector. Thus it is shown that a reasonable and practical detector when little information is known about the phase noise is the random step detector. If however the phase noise has a linear drift characteristic which is known statistically then such information could be used to design a more efficient detector, the random linear drift detector.

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FROM: L. Schuchman

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### TECHNICAL MEMORANDUM

This paper extends the investigation of lunar-Earth, planetary-Earth and deep space low capacity channels initiated in a previous paper by the author.<sup>1</sup> In that paper the optimum maximum likelihood detector for a signal distorted by phase noise  $\theta_i(t)$  was described and defined as the random step detector.\* The variate  $\theta_i(t)$  is a stepped approximation to the actual phase noise and is illustrated in Figure 1.

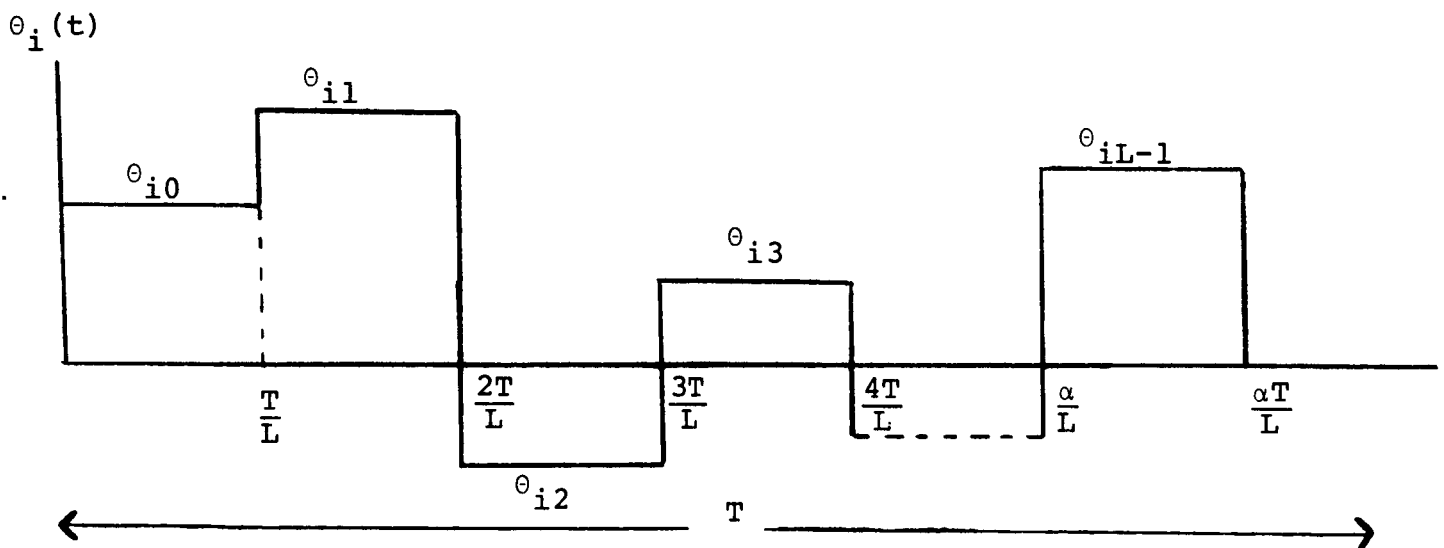


Figure 1 - The Random Phase as a Function of Time

\*The modulation assumed was M'ary FSK so that  $i=1, 2, \dots, M$ .

The key assumption made was that the set  $\{\theta_{ij}\}$  variates are independent and identically distributed. Section II of this memorandum is a validation of this assumption.

Section III describes a second maximum likelihood detector which is defined as the random linear drift detector. The optimality of this second receiver depends upon a greater a priori knowledge of the statistical characteristics of the phase noise than is assumed for the model previously described.

## II. THE RANDOM STEP DETECTOR - INDEPENDENT $\theta_{ij}$ 's

Intuitively one feels that the set  $\{\theta_{ij}\}$  must be correlated. However any attempt to model the phase noise with a correlated set of  $\{\theta_{ij}\}$  results in a mathematical complexity which implies an extremely difficult detector realization. Thus attempts to model  $\theta_i(t)$  as a first order Markov process generally do not lead to satisfactory results. This can be seen in Appendix A where such a Markov model is described.

Since one cannot model the phase noise assuming correlation in the set of  $\theta_{ij}$ 's, and derive a realizable receiver and since it is believed that the  $\theta_{ij}$ 's are indeed correlated, of what value is the derived detector for which an independent set of  $\theta_{ij}$ 's was assumed? To answer this question one can logically argue in the following manner.

If phase noise exists and its bandwidth (B) is measurable, then the minimum number of variates in the set  $\{\theta_{ij}\}$  is approximately determined by such a measurement. Thus since j varies from 1 to L-1 the minimum value of L is determined by  $L_{\text{minimum}} = \lceil \frac{1}{TB} \rceil$  where T is the time duration of a transmitted M'ary symbol and B is the rf bandwidth. Thus any increase in L, or corresponding widening of the detection bandwidth, would not result in any improvement in the detection of the phase noise signal but would rather lead to a degradation in performance. This is analogous to the use of frequency diversity to combat a multiplicative noise in that an optimum diversity value (maximum value for transmission bandwidth) exists which minimizes the effect of fading. Any additional increase

in diversity results in a degradation caused by the increase in additive thermal noise accompanying the increase in the transmission bandwidth. This phenomena is depicted in Figure 2.

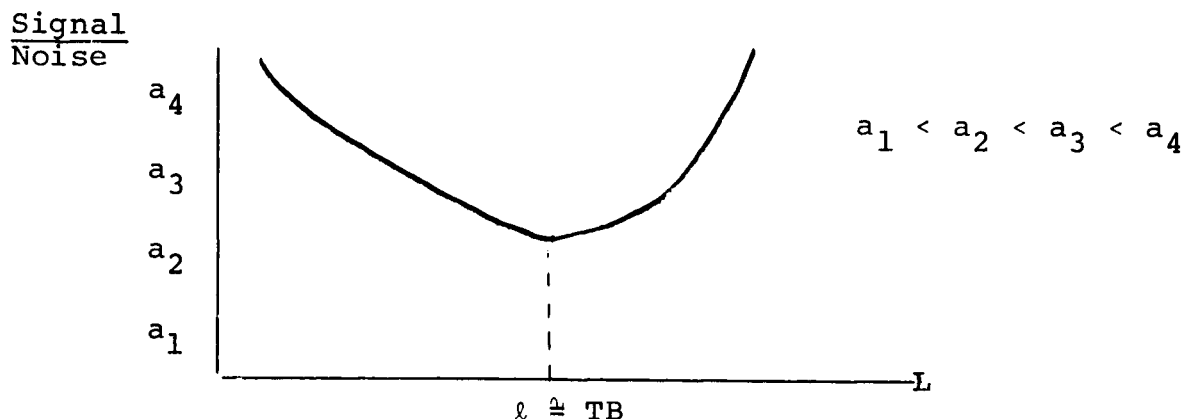


Figure 2 - Variation of the ratio of received signal to noise power with  $L$  (probability of bit error = constant)

In addition note that as the phase noise bandwidth increases the signal is increasingly distorted and the value of  $\ell$  increases so that the performance is degraded. This is graphically depicted in Figure 3.

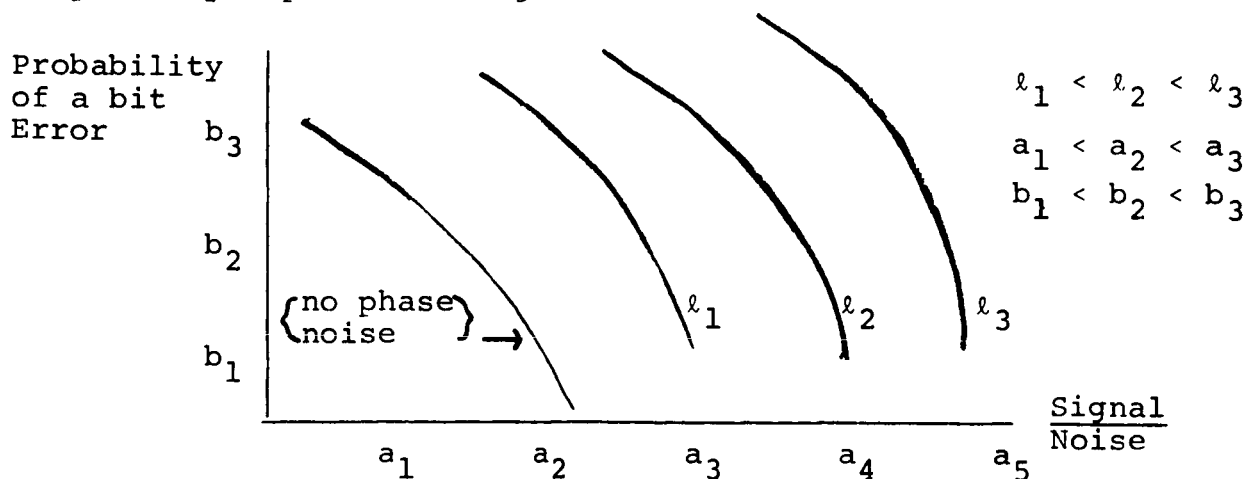


Figure 3 - Relative Performance of the Random Step Detector

If one refers to the previous companion memorandum, it is seen that each of the set of  $i$  measures ( $i=1,2,\dots,M$ ) is computed from the weighting of  $L$  noncoherent matched filter energy measures, each having been integrated over a mutually exclusive time interval  $\tau$  (where  $L\tau = T$ ). The  $i$  measures are then compared with a decision made that signal  $k$  was transmitted if the  $k^{\text{th}}$  measure is the largest. Now if such a receiver is used and a signal is transmitted in which the set of  $\{\theta_{ij}\}$  variates are not independent the performance of the receiver will be identical to one where the set of  $\{\theta_{ij}\}$  variates are truly independent. This is easy to show since each of the  $L$  energy measures computed and weighted to form each of the  $M$  compared decision measures is independent of the phase information (as are the weighting factors).<sup>\*</sup> Thus one can conclude that this receiver acts as an upper bound in performance since if one could realize a receiver for a correlated set of  $\theta_{ij}$ 's then one would expect such a receiver to make use of such correlation information and to perform better than a receiver which throws such information away. (By the same reasoning one would expect a coherent receiver to perform better than a noncoherent receiver in additive white Gaussian noise and it does.)

It is concluded, therefore, that if the exact phase noise structure is not known but only the phase noise bandwidth prudence dictates that one design a receiver that does not require the phase noise structure for its operation and whose performance we can predict. In addition it is equally reasonable to use the receiver described in Reference 1 since it is the optimum of such realizable receivers.

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<sup>\*</sup>Note that not all noncoherent detectors eliminate phase information. For example if an optimum noncoherent detector (thermal noise only) were used, and the signal transmitted did in fact possess phase noise the compared energy measures would in this case be functions of the phase noise parameters. Thus the performance of the detector under thermal noise conditions would not bound the performance when the signal was corrupted by both thermal and phase noise.

This section is ended with the following\* observation. The quadrature detector has been shown by Ferguson<sup>2</sup> to be in the limit the optimum detector for small predetection signal to noise ratios for several different models of the phase noise. However the generalized form of the quadrature detector is such that M decision variables  $\{z_j\}$  are formed by obtaining

$$z_j = \int_{-\infty}^{+\infty} a(\omega) p(\omega - \omega_j) d\omega \quad (1)$$

where  $j = 1, 2, \dots, M$ .

$a(\omega)$  is a spectral coefficient defined by

$$a(\omega) = \left\{ \int_0^T \tilde{y}(t) \cos \omega t dt \right\}^2 + \left\{ \int_0^T \tilde{y}(t) \sin \omega(t) dt \right\}^2$$

$\tilde{y}(t)$  is the received signal plus noise, i.e.,  $\tilde{y}(t) = x(t) + n(t)$ ;  $\omega$  is the frequency at which  $a(\omega)$  is evaluated. The signal is  $x(t)$  and  $n(t)$  is white Gaussian noise.

$p(\omega - \omega_j)$  is a specific spectral weighting function centered about the frequency  $\omega_j$  the  $j^{\text{th}}$  tone.

T is the length of a transmitted M'ary symbol.

Now note the following. If  $p(\omega - \omega_j) = \delta(\omega - \omega_j)$ , where  $\delta(\omega)$  is the Dirac delta function, equation (1) reduces

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\*It was shown that for many real applications very small predetection signal-to-noise ratios may be an unrealistic assumption to make about a low capacity channel.

to the standard optimal noncoherent receiver for the no phase noise case. Thus the structure of the receiver in the presence of phase noise is dependent on the  $p(\omega - \omega_j)$  weighting function which in turn is dependent upon the knowledge of the noise. If little is known about the noise it is reasonable to assume  $p(\omega)$  to be uniformly distributed and this is precisely what has been done.<sup>2</sup> Thus an analogy in assumptions between the correlation of the  $\theta_{ij}$ 's for the random step detector model and the form of the weighting function  $p(\omega - \omega_j)$  for the spectrum analyzer exists.

### III. THE RANDOM LINEAR DRIFT DETECTOR

In the last section it was argued that if only the phase noise bandwidth is known, the use of the optimum conservative detector is the most reasonable approach to take.

There may exist however more information about the phase noise than just its bandwidth. In such cases the design of a receiver based upon a random step approximation to the noise would be overly conservative. Unfortunately, as has been shown in Appendix A, knowledge of the noise cannot easily lead to a practical realization of the maximum likelihood receiver. However there are a few exceptions. Modeling the phase noise as a random linear drift detector is one such exception and is developed in this section.

The phase noise for this model is assumed to have a linear drift rate over a transmission symbol period  $T$ . What is not known is from what point in the phase domain the drift starts in each transmission symbol. The drift rate is also assumed unknown. A sample waveform of such a process is shown in Figure 4.

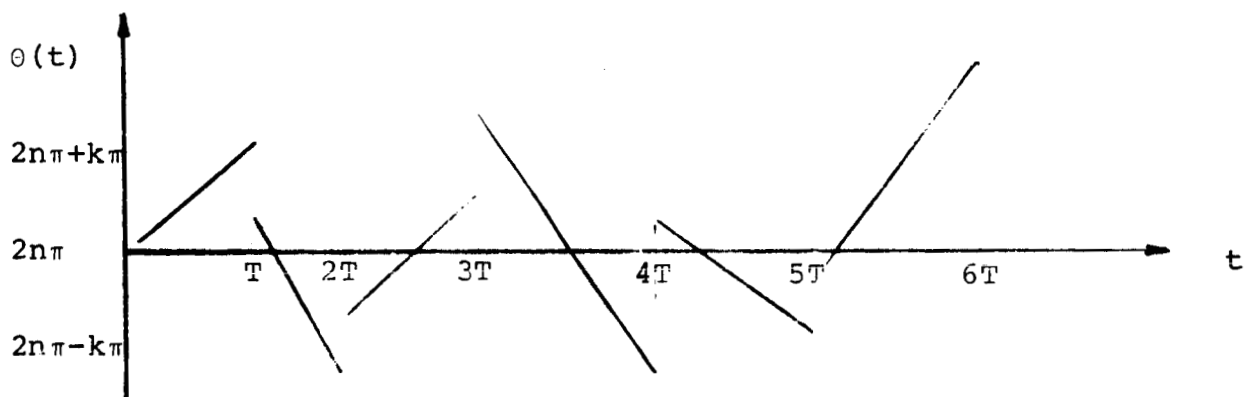


Figure 4 - The Drift Phase Noise Model

The detector for the phase noise model described in Figure 4 is derived in Appendix B\*, under the assumption that the drift rate is known except for sign information. The resultant detector computes M measures  $\{\gamma_k\}$  and decides that the signal transmitted is

$$\max\{\gamma_k\} \quad k=1,2,\dots,M$$

where  $\gamma_k$  is given by

$$\gamma_k = {}_1\gamma_k + {}_2\gamma_k \quad (2)$$

and where

$${}_i\gamma_k = \begin{cases} M(z_{ki}) & z_{ki} \leq 3.75 \\ N(z_{ki}) & z_{ki} > 3.75 \end{cases} \quad i=1,2$$

$$\begin{aligned} M(z_{ki}) = & 1 + 3.5156229\eta^2 + 3.0899424\eta^4 + \\ & 1.2067492\eta^6 + 2.659732\eta^8 + \\ & .0360768\eta^{10} + .0045813\eta^{12} \end{aligned}$$

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\*In Appendix B we make use of a stepped approximation to the slope assuming that every  $\tau$  seconds the slope increases by a factor  $\Delta\theta$ .

$$N(z_{ki}) = \left[ .3984228 + .03988024\eta^{-1} - .00362018\eta^{-2} + \right. \\
.00163801\eta^{-3} - .01031555\eta^{-4} + .02282967\eta^{-5} \\
\left. .02895312\eta^{-6} + .01787654\eta^{-7} - .00420059\eta^{-8} \right].$$

$$\frac{e^{z_{ki}}}{z_{ki}^{1/2}}$$

where

$$\eta = z_{ki}/3.75$$

$z_{ki}^2$  is given by

$$z_{ki}^2 = \left[ \sum_{j=1}^L \left[ \alpha_{k1j} \cos(j-1)\Delta\theta + (-1)^i \alpha_{k2j} \sin(i-1)\Delta\theta \right] \right]^2 + \\
\left[ \sum_{j=1}^L \left[ (-1)^{2i+1} \alpha_{k1j} \sin(j-1)\Delta\theta + \alpha_{k2j} \cos(j-1)\Delta\theta \right] \right]^2 \quad (3)$$

and where

$$\alpha_{k1j} = \int_{(j-1)\tau}^{j\tau} y(t) \cos w_k t \, dt \\
\alpha_{k2j} = \int_{(j-1)\tau}^{j\tau} y(t) \sin w_k t \, dt \quad (4)$$

$y(t)$  is the received signal and  $w_k$  is the transmission frequency corresponding to the  $k^{\text{th}}$  symbol  $k=1,2,\dots,M$ .

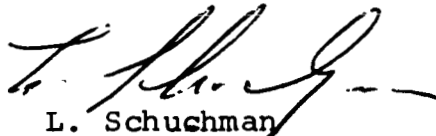
Thus the detector generates  $M$  quadrature pairs of a noncoherent matched filter every  $\tau$  seconds for  $L$  such intervals, and weights them with the proper  $\frac{\sin}{\cos} \{(j-1)\Delta\theta\}$  values to form the  $z_{ki}$  parameters which are in turn weighted to produce the  $M \gamma_k$  measures used to determine which  $M$ 'ary symbol was transmitted.

The question remains as to how to determine  $L$ . It would appear that the smaller you make  $L$  the better is the approximation to the linear slope model. Since the linear slope model is just a model, and since practical limitations of implementation must be considered, the value of  $L$  would have to be determined experimentally.

The assumption of a near constant known slope (except for its sign) appears to be unrealistic in that if one could know such information then one would use a coherent detector. To extend the result it is assumed that the drift of the receiver is random, or in our model of a stepped approximation to the slope, that  $\Delta\theta$  can vary from  $-\pi$  to  $\pi$ . Then we need only require that  $\Delta\theta$  be known statistically rather deterministically. Thus if the density distribution of  $\Delta\theta$  for a given value of  $L$  ( $p(\Delta\theta/L)$ ) could be determined experimentally then using equation (2) the  $\gamma_k(L)$  measures are given by:

$$\gamma_k(L) = \int_0^\pi \left[ {}_1\gamma_k(\Delta\theta, L) + {}_2\gamma_k(\Delta\theta, L) \right] p(\Delta\theta/L) d(\Delta\theta) \quad (5)$$

Although this is difficult to do analytically it is easy to do numerically.

  
L. Schuchman

2034-LS-jf

Attachments  
References  
Appendices A and B

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REFERENCES

1. L. Schuchman, "Optimization of a Very-Low Capacity Channel Using a Multi-Tone Frequency Shift Keyed Detector", Bellcomm Technical Memorandum, TM-69-2034-4, May 5, 1969.
2. M. J. Ferguson, "Communication at Low Data Rates", Spectral Analysis Receivers", IEEE Trans. of Communication Technology, Vol. Com-16, October 1968.

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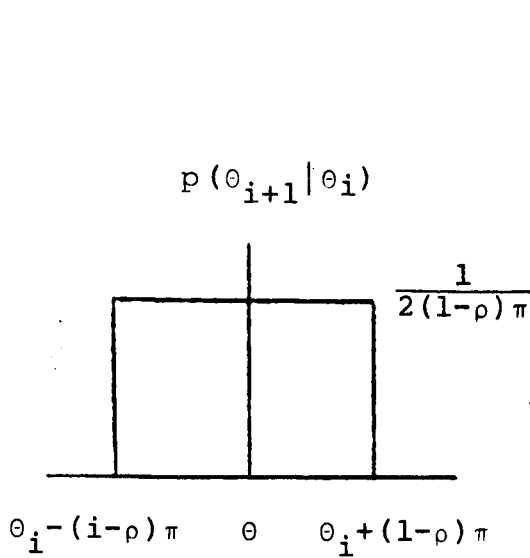
## APPENDIX A

### PHASE NOISE MODEL I - NONINDEPENDENT $\theta_{ij}$ 's

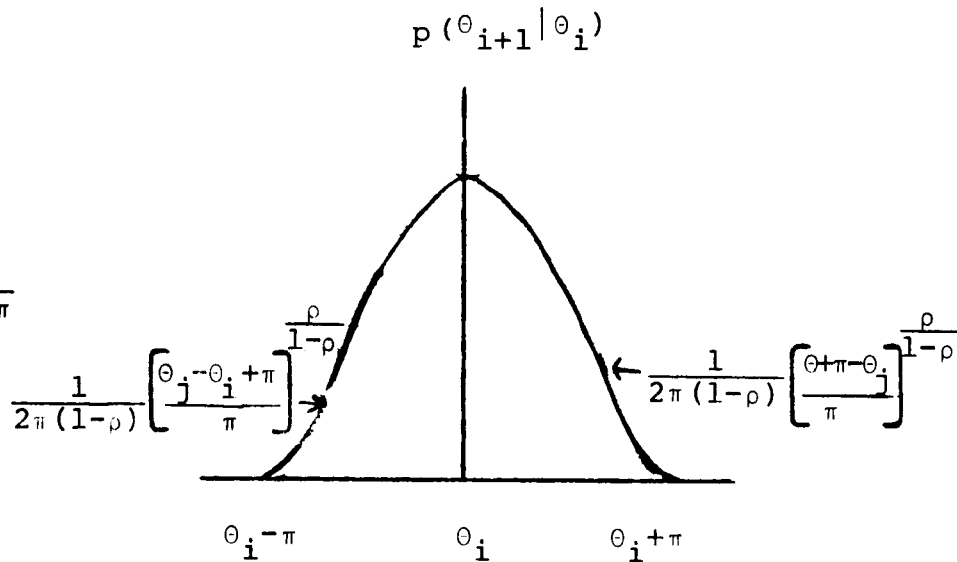
The joint density distribution for the  $\theta_{ij}$ 's will be assumed to exhibit a first order Markov property so that we may write

$$p(\theta_1, \theta_2, \theta_3, \dots, \theta_i) = p(\theta_1) p(\theta_2/\theta_1) p(\theta_3/\theta_2), \dots, p(\theta_i/\theta_{i-1}) \quad (A-1)$$

This is a natural and simple extension of the model which assumes the set of  $\theta_{ij}$ 's are independent. Two models of the conditional density  $p(\theta_{i+1}/\theta_i)$ ,  $i=1,2,\dots,L-1$  are used. These are described below.



Model A-1



Model A-2

where  $(0 \leq \rho \leq 1)$ .

$\theta_1$  is assumed to be uniformly distributed over a  $2\pi$  interval.

The procedure followed in the derivation parallels the one used in Appendix A of Reference 1 so that only a minimum number of statements are made in this derivation. It is thus easy to show that one need calculate M conditional probability density distributions  $p[y(t)|\omega_\ell]$  where  $p[y(t)|\omega_\ell]$  is the probability density of the received signal  $y(t)$  assuming that signal  $\ell$ , ( $\ell=1,2,\dots,M$ ) was transmitted. This probability can be written conditionally as

$$p[y(t)|\omega_\ell] = \int_{\{\theta_i\}} p(y(t)|\omega_\ell, \theta_1, \theta_2, \dots, \theta_L) \cdot p(\theta_1) \cdot \quad (A-2)$$

$$p(\theta_2/\theta_1), \dots, p(\theta_L/\theta_{L-1}) d\{\theta_i\}$$

Letting  $L=2$  and using model A-1 and the assumption that  $y(t)=x(t)+n(t)$  where  $x(t)$  is the signal and  $n(t)$  is white Gaussian noise leads to

$$p(y(t)|\omega_\ell) = \int_{-\pi}^{+\pi} \int_{\theta_1 - \pi(1-\rho)}^{\theta_1 + \pi(1-\rho)} e^{-\alpha_{1\ell} \cos(\theta_1 + \beta_1) - \alpha_{2\ell} \cos(\theta_2 + \beta_2)} \frac{d\theta_1 d\theta_2}{2\pi(1-\rho)} \quad (A-3)$$

where

$$\alpha_{i\ell} = \frac{2}{N_0} \left( \frac{2E}{T} \right)^{1/2} \left[ \alpha_{i\ell 1}^2 + \alpha_{i\ell 2}^2 \right]^{1/2}$$

$$\beta_i = \tan^{-1} \frac{\alpha_{i2\ell}}{\alpha_{i1\ell}}$$

the  $\alpha_{ij\ell}$ 's are defined in equation (4) of the paper.

Expanding  $\exp[-\alpha_{2\ell} \cos(\theta_{2\ell} + \tau_{2\ell})]$  in a Bessel series and integrating and then expanding  $\exp[-\alpha_{1\ell} \cos(\theta_{1\ell} + \tau_{1\ell})]$  in a similar Bessel series, rearranging terms, and integrating leads to

$$p(y(t) | \omega_{\ell}) = I_0(\alpha_{1\ell}) I_0(\alpha_{2\ell}) + 2 \sum_k I_k(\alpha_{1\ell}) I_k(\alpha_{2\ell}) \left[ \sin \left[ \frac{k\pi(1-\rho)}{k\pi(1-\rho)} \right] \right] \cdot \cos k(\beta_{2\ell} - \beta_{1\ell}) \quad (A-4)$$

where  $I_k(\beta)$  is a modified Bessel function of the  $k^{\text{th}}$  order and of argument  $\beta$ .

In a similar manner we find that for model A-2  $p(y(t)/\omega_{\ell})$  is given by

$$p(y(t)/\omega_{\ell}) = I_0(\alpha_{1\ell}) I_0(\alpha_{2\ell}) + \frac{2}{\pi(1-\rho)} \sum_{k=1}^{\infty} I_k(\alpha_{1\ell}) I_k(\alpha_{2\ell}) [\cos k(\beta_{2\ell} - \beta_{1\ell})] \cdot b(\rho) \quad (A-5)$$

where

$$b(\rho) = \int_0^{\pi} \left( \frac{u}{\pi} \right)^{\rho/1-\rho} \cos k u \, du$$

Note that for both models when  $\rho=0$

$$p(y(t)/\omega_\ell) = I_0(\alpha_{1\ell}) I_0(\alpha_{2\ell}) \quad (A-6)$$

and when  $\rho=1$

$$p(y(t)/\omega_\ell) = I_0(\alpha_\rho) \quad (A-7)$$

where

$$\alpha_\ell^2 = \left\{ \int_0^T y(t) \cos \omega_\ell t dt \right\}^2 + \left\{ \int_0^T y(t) \sin \omega_\ell t dt \right\}^2$$

To derive equation (A-7) we used the identity

$$I_0(W) = I_0(z_1) I_0(z_2) + 2 \sum_{n=1}^{\infty} (-1)^n I_n(z_1) I_n(z_2) \cos n\phi$$

where

$$W = (z_1^2 + z_2^2 - 2z_1 z_2 \cos \phi)^{1/2}$$

If the value of  $\rho$  lies between zero and one the infinite summations in both equations (A-4) and (A-5) do not reduce. Thus even for the case of  $L=2$  the receiver is quite complex. It is easy to show that this complexity increases nonlinearly with  $L$  so that realization of a detector for  $L=3$  appears impossible for arbitrary values of  $\rho$ .

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## APPENDIX B

### PHASE NOISE MODEL II

( $\Delta\theta$  assumed known)

In this Appendix as in the previous one we have that  $\theta_i(t)$  is a first order Markov process so that

$$p(\theta_1, \theta_2, \dots, \theta_L) = p(\theta_1) p(\theta_2/\theta_1), \dots, p(\theta_L/\theta_{L-1})$$

where

$$p(\theta_i/\theta_{i-1}) = \delta(\theta_i - (\theta_{i-1} + \Delta\theta(\text{sgn}\Delta\theta))) p(\text{sgn}\Delta\theta)$$

$$i = 3, 4, \dots, L$$

$$p(\theta_2/\theta_1) = \delta(\theta_2 - (\theta_1 + \Delta\theta(\text{sgn}\Delta\theta))) p(\text{sgn}\Delta\theta)$$

$$p(\text{sgn}\Delta\theta) = \frac{1}{2} \delta(\text{sgn}\Delta\theta - 1) + \frac{1}{2} \delta(\text{sgn}\Delta\theta + 1)$$

and  $\theta_1$  is assumed to be uniformly distributed.

Thus the conditional densities are given by

$$\begin{aligned}
 p(y(t) | \omega_\ell) = & \int_{-\pi}^{\pi} \left[ \exp \left\{ - \sum_{i=1}^{\ell} \alpha_{i\ell} \cos \left[ \beta_{i\ell} + \text{sgn} \Delta \theta [i-1] \Delta \theta \right] \cos \theta_1 + \right. \right. \\
 & \left. \left. \sum \alpha_{i\ell} \sin \left[ \beta_{i\ell} + [\text{sgn} \Delta \theta] [i-1] \Delta \theta \right] \right\} \right] \\
 & \cdot \left[ \frac{\delta(\text{sgn} \Delta \theta - 1)}{2} + \frac{\delta(\text{sgn} + 1)}{2} \right] d\theta_1 d\text{sgn} \Delta \theta
 \end{aligned} \tag{B-1}$$

which can be written as

$$p(y(t) | \omega_\ell) = \frac{I_0(z_{\ell 1}) + I_0(z_{\ell 2})}{2} \tag{B-2}$$

where  $z_{\ell i}^2$  is given by equation (3) of the text.

Expanding the Bessel function in a polynomial as described in Reference 1 leads to equation (2) of the text.